Sample results are given for $E = 8.8$, $\kappa = 0.4$ and $\Pr_\epsilon = 1$. Velocity profiles for an adiabatic stationary wall and various Mach numbers are shown in Fig. 2 along with low-speed experimental data of Robertson[2] with which the calculated $M_\infty = 0$ profile compares favorably. The difference in Re_d from 10^5 to 0.6×10^5 (that of the data taken from Ref. 2) accounts for such a small shift in the velocity profile that it does not affect the comparison.

The influence of the stationary wall enthalpy on skin-friction and heat-transfer coefficients for $M_\infty = 4$ and various Reynolds numbers is shown in Fig. 3. The heat rates at both walls can be calculated from the relations

$$q_w = C_H\rho_\infty U_\infty(h_w - h_r) = -\tfrac{1}{4}\bar{q}C_f\rho_\infty u_\infty^3,$$

$$q_\infty = \tfrac{1}{2}(1 - \bar{q}/2)C_f\rho_\infty u_\infty^3$$

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Numerical Quadrature and Radiative Heat-Transfer Computations

D. C. Look Jr.*

University of Missouri, Rolla, Rolla, Mo.

AND

T. J. Love†

University of Oklahoma, Norman, Okla.

Nomenclature

$B(x),B(y)$	= nondimensional radiosity of one variable
$B(x,y)$, $B(\xi,\eta)$	= nondimensional radiosity of two variables
$B_r(x)$	= convergent nondimensional radiosity value from Ref. 2 for infinite adjoint plates
x,y,ξ,η	= normalized independent variable
ρ	= reflectance
$K(x,y)$, $K(x,y,\xi,\eta)$	= kernels of the integral equations
W_i	= weighting coefficients of the numerical quadrature method
Z_1	= artificial parameter for one variable geometry
Z_2	= artificial parameter for two variable geometry
n	= order of quadrature or number of intervals associated with a numerical quadrature method
θ	= angular separation of infinite adjoint plates
E	= approximate percentage error between $B(x)$ and $B_r(x)$

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* Assistant Professor, School of Mechanical and Aerospace Engineering. Member AIAA.

† Director, School of Aerospace and Mechanical Engineering. Member AIAA.

THE integral equations involved in radiative heat-transfer computations usually do not lend themselves to exact analytical solutions. Numerical quadrature are thus often used to obtain approximate solutions. The question of the validity of the quadrature selected must be considered. That is, some precaution must be taken in applying a numerical quadrature to the integral equation involved.

Consider the nondimensional form of the equation for the radiosity of a diffuse gray enclosure in terms of a single variable (Ref. 1, p. 81)

$$B(x) = 1 + \rho \int B(y)K(x,y)dy \tag{1}$$

An approximate solution may be obtained by the substitution of numerical quadrature for the integral in Eq. (1). Thus

$$B(x_i) = 1 + \rho \sum_{j=0}^{n} W_j B(x_j)K(x_i,x_j) \tag{2}$$

Equation (2) may be rearranged to the form

$$1 = [1 - \rho W_i K_{ii}]B_i - \rho \sum_{\substack{j=0\\ i\neq j}}^{n} B_j K_{ij} W_j \tag{3}$$

where $B_i = B(x_i)$, $K_{ij} = K(x_i,x_j)$. Since all the quantities in Eq. (3) are positive, mathematically plausible approximate solutions are available if and only if

$$Z_1 = 1 - \rho K_{ii} W_i > 0 \tag{4}$$

If, for example, the trapezoidal rule is to be used, $W_o = W_n = 1/2n$ and $W_1 = \ldots = W_{n-1} = 1/n$ where $x_i = -\frac{1}{2} + i/n$, $0 \le i \le n$. It is easily seen that as n increases, W_i approaches zero and depending on the form of K_{ii}, Eq. (4) may become valid and Z_1 will approach one. Applying this example to the problem of plane parallel plates of infinite length (Fig. 1a), Eq. (4) takes the form

$$Z_{11} = 1 - \rho/2nH > 0 \tag{5}$$

Also applying this example to plane adjoint plates of infinite length (Fig. 1b), Eq. (4) takes the form

$$Z_{12} = 1 - (\rho/4nx_i)\cot(\theta/2)\cos(\theta/2) > 0 \tag{6}$$

Notice that Eq. (5) indicates the selection of n is such that $n > \rho/2H$, while Eq. (6) indicates $n > (\rho/4x_i)\cot(\theta/2)\cos(\theta/2)$. Thus, the trapezoidal rule may be easily applied to the parallel plates case, but will introduce large error in the adjoint plates case.

Gaussian quadrature may be applied to the adjoint plates case but only very carefully as may be seen by Table 1. The parameter Z has been included in this table to illustrate its usefulness. It may be noted that as the parameter Z approaches one, the approximate percentage error approaches zero. This indicates that if the parameter Z is greater than zero, no negative radiosities occur and as the value of

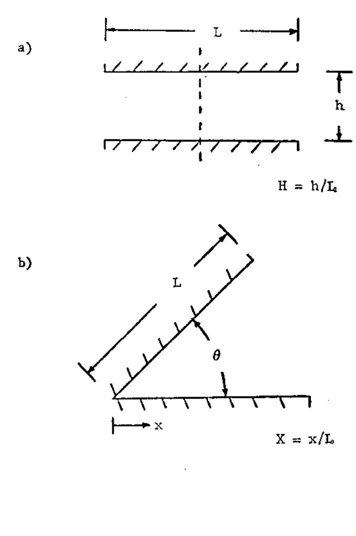

Fig. 1 Geometry of infinite length plates: a) parallel and b) adjoint.

Table 1 Nondimensional radiosity at the first and second gaussian quadrature points for the case of infinite adjoint plates

ρ	θ	$X = x/L$	$B(x)$ $n = 3$	$B(x)$ $n = 15$	Z	$B_r(x)$	E, %
0.1	90	0.1127	1.0576		0.9565	1.0463	1.07
		0.5000	1.0282		0.9843	1.0285	0
		0.0060		1.0683	0.9548	1.0522	1.14
		0.0314		1.0506	0.9802	1.0508	0
	60	0.1127	1.1155		0.9077	1.0753	3.74
		0.5000	1.0524		0.9667	1.0526	0
		0.0060		1.1237	0.9042	1.0807	3.97
		0.0314		1.0814	0.9580	1.0795	0
	45	0.1127	1.1708		0.8628	1.0850	8.84
		0.5000	1.0711		0.9491	1.0683	0
		0.0060		1.1807	0.8575	1.0931	8.00
		0.0314		1.1006	0.9375	1.0920	0
0.5	90	0.1127	1.3613		0.7825	1.2757	6.72
		0.5000	1.1629		0.9215	1.1620	0
		0.0060		1.4218	0.7742	1.3262	7.20
		0.0314		1.3129	0.9010	1.3129	0
	60	0.1127	2.0040		0.5387	1.5205	31.90
		0.5000	1.3479		0.8335	1.3335	1.08
		0.0060		2.1457	0.5210	1.5868	35.20
		0.0314		1.6015	0.7900	1.5706	1.90
	45	0.1127	3.4204		0.3144	1.6725	104.00
		0.5000	1.5429		0.7456	1.4644	5.10
		0.0060		3.8447	0.2879	1.7318	122.00
		0.0314		1.8626	0.6878	1.7184	8.40
0.9	90	0.1127	1.8659		0.6086	1.6131	15.70
		0.5000	1.3521		0.8587	1.3403	0.88
		0.0060		2.0818	0.5935	1.7718	17.50
		0.0314		1.7326	0.8218	1.7265	0.35
	60	0.1127	7.2931		0.1697	2.4867	>200.00
		0.5000	2.2249		0.7003	1.8364	20.60
		0.0060		10.3946	0.1378	2.8729	>200.00
		0.0314		3.3463	0.6220	2.7554	21.40

Z approaches one the error between the approximate solutions and the convergent solutions approaches zero.

Application of this analysis to problems of two variables follows immediately. That is, for a gray enclosure

$$B(x,y) = 1 + \rho\iint B(\xi,\eta)K(x,y,\xi,\eta)d\xi d\eta$$

which may be approximated by

$$B(x_i,x_j) = 1 + \rho \sum_{1} \sum_{k} B(x_1,x_k)K(x_i,x_j,x_1,x_k)W_1W_k$$

Rearrangement and application of the same line of reasoning yields

$$Z_2 = 1 - \rho W_iW_jK(x_i,x_j,x_i,x_j) > 0 \tag{7}$$

Application of various quadrature formulas to Eq. (7) for the case of finite parallel and adjoint plates indicates that the parallel plates case may be easily solved by any quadrature method, but that not even gaussian quadrature may be applied to the adjoint plates case with any reliability.

If the parameter Z is less than zero, Eqs. (3) or its equivalent for the two variable case are impossible from a physical standpoint. Thus, an apparent singular point of the integral is not adequately approximated by numerical quadrature. The results of this study seems to imply that the numerical quadrature approximation tends to over estimate the kernel of the integral equation for points directly opposed which are near the apparent singularity of the integral.

Thus, parameters have been introduced which yield information essential to the use of numerical quadrature as a method of solution of integral equations encountered in radiative heat-transfer computations. These parameters should approach one as the order of quadrature or the number of intervals is increased before valid approximate solutions may be obtained.

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Maximum Errors from Quantization in Multirate Digital Control Systems

Charles L. Phillips* and John C. Johnson†
Auburn University, Auburn, Ala.

A PROBLEM in the implementation of digital control systems is the determination of the number of bits required to adequately represent signals within the digital controller. The quantization of signals within the controller introduces errors into the system output, and the magnitudes of the errors are a direct function of the number of bits used to represent signals within the controller. Techniques have been developed which yield both the mean-square error and an upper bound on the errors for single rate systems,[1–3] and that yield the mean-square error for a multirate system.[4] This paper presents a technique for determining an upper bound on the system errors due to quantization in a multirate controller.

Multirate System

Consider the multirate control system shown in Fig. 1. The z-transform variables are defined as $z = \epsilon^{sT}$, and $z_n = z^{1/n}$. For the system

$$Y(z) = \{\mathfrak{z}[D(z_n)G(z_n)]R(z)\}/\{1 + \mathfrak{z}[D(z_n)G(z_n)]\} \tag{1}$$

In Eq. (1),

$$G(z_n) = \mathfrak{z}n\{(1 + \epsilon^{-Ts/n})\ G(s)/s\} \tag{2}$$

i.e., the pulse transfer function of the data-hold and plant combination, with sampling period T/n. For notation, let

$$G(z_n)|_{z_n = z^{1/n}} = G(z)_n$$

Now let

$$G_f(z)_n = D(z)_nG(z)_n = g_f(0) + g_f(1/n)z^{-1/n} + \ldots + g_f(1)z^{-1} + \ldots \tag{3}$$

Then the z-transform of $D(z)_nG(z)_n$ is

$$\mathfrak{z}[D(z)_nG(z)_n] = \mathfrak{z}[D(z_n)G(z_n)] = g_f(0) + g_f(1)z^{-1} + g_f(2)z^{-2} + \ldots \tag{4}$$

It is desired now to investigate the errors that appear in the output due to quantization within the digital controller. The digital controller transfer function, $D(z_n)$, is realized by some

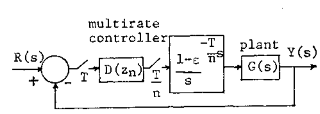

Fig. 1 Multirate control system.

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* Professor.
† Research Associate.